Applicative Intersection Types

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New languages keep being invented!

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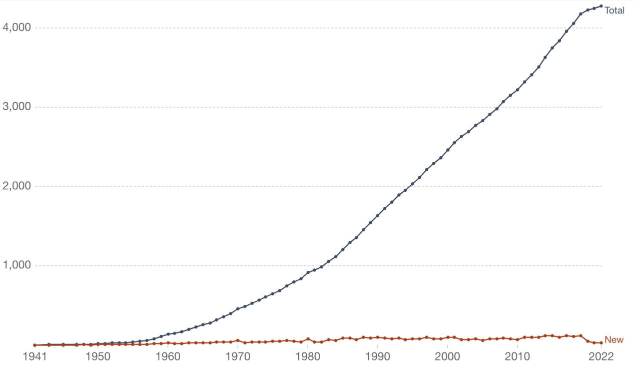

credit: pldb.com

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New features keep being discovered!

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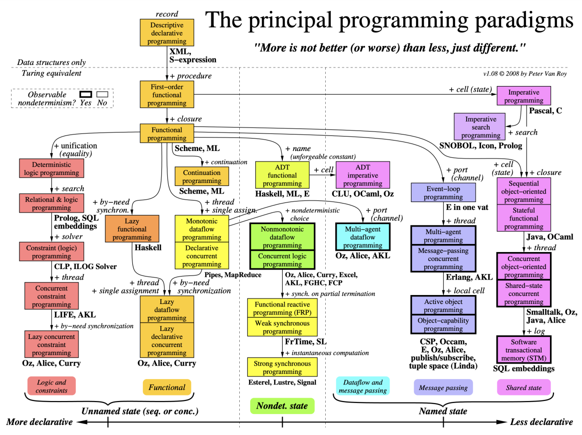

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- New languages are prototyped *by a small/core calculus*;
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- New languages are prototyped *by a small/core calculus*;
- New features are often studied *in an isolated environment*;

And...

- Features are not orthogonal;
- Languages are not designed at once.

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- contains wide features;
- retains simplicity;
- has extensibility;
- and enjoys good properties,

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- contains wide features;
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- has extensibility;
- and enjoys good properties,

is *desired* by language designers and implementors.

Intersection Types is a nice fit

*"our goal is to use **intersections** and unions as general mechanisms for encoding language features, so we really should do it in full generality, or not at all..."*[1]

[1] Jana Dunfield. "Elaborating intersection and union types". In: *Journal of Functional Programming* 24.2-3 (2014), pp. 133–165.

Intersection Types

- A term e having the type $A \& B$ means e has both A and B.

[2] Mario Coppo, Mariangiola Dezani-Ciancaglini, and Betti Venneri. "Functional characters of solvable terms". In: *Mathematical Logic Quarterly* 27.2-6 (1981), pp. 45–58.

Intersection Types

- A term e having the type $A \,\&\, B$ means e has both A and B.
- Originally introduced by Coppo et al.[2], it allows $\lambda x.\, x\, x$ to be typed $((A \to B) \,\&\, A) \to B$.

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- Originally introduced by Coppo et al.[2], it allows $\lambda x.\, x\, x$ to be typed $((A \to B) \mathbin{\&} A) \to B$.
- In languages like TypeScript, the intersection types are explicitly inhabitated.

```
interface Name { name: string; }
interface ID { id: number; }
type Person = Name & ID
let e : Person = { id: 42, name: 'Alice'};
```

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Merge Operator[4]

- $e_1 \,,\, e_2$ means it can be used as e_1 or e_2.

[3] Bidirectional typing, $\Gamma \vdash e \Leftrightarrow A$, and $\Leftrightarrow ::= \Leftarrow | \Rightarrow$. $\Leftarrow$ is to check; $\Rightarrow$ is to infer.

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- $e_1 ,, e_2$ means it can be used as e_1 or e_2.
- Force intersection types to be *explicitly* introduced and inhabitated.
- Typing for merge is [3]

$$\frac{\Gamma \vdash e_1 \Rightarrow A \qquad \Gamma \vdash e_2 \Rightarrow B}{\Gamma \vdash e_1 ,, e_2 \Rightarrow A \,\&\, B}\ \textsc{T-Mrg}$$

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- Merge operator adds expressive power and enables many applications.

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Extensible Records[5]

- Records can be represented by *syntactic sugar of merge operator*.
- $\{x = e_1, y = e_2, z = e_3\}$ can be viewed as $\{x = e_1\}, , \{y = e_2\}, , \{z = e_3\}$.

[5] Luca Cardelli and John C Mitchell. "Operations on records". In: *Mathematical structures in computer science* 1.1 (1991), pp. 3–48.

Extensible Records[5]

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- $\{x = e_1, y = e_2, z = e_3\}$ can be viewed as $\{x = e_1\}, , \{y = e_2\}, , \{z = e_3\}$.
- Record width subtyping *for free*.

$$\{l_i : T_i\}^{i=1..n..n+k} <: \{l_i : T_i\}^{1..n}$$

is subsumed by

$$\{l_1 : A\} \,\&\, \{l_2 : B\} <: \{l_1 : A\}$$

is subsumed by

$$A \,\&\, B <: A$$

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Record Projection

- Record Projection is standard.

$$(\{x = e_1\} ,, \{y = e_2\}).x \hookrightarrow e_1$$

$$(\{x = e_1\} ,, \{y = e_2\}).y \hookrightarrow e_2$$

- Record Concatenation is simply merging.

$$(\{x = e_1\} ,, \{y = e_2\}) ,, \{z = e_3\}$$

Overloaded Functions[6]

- Function implementation *varies* depending on the types of arguments.

[6] Giuseppe Castagna, Giorgio Ghelli, and Giuseppe Longo. "A calculus for overloaded functions with subtyping". In: *Information and Computation* 117.1 (1995), pp. 115–135.

Overloaded Functions[6]

- Function implementation *varies* depending on the types of arguments.
- Consider Haskell's `show` function.

```
show :: Show a => a -> String
instance Show Int where
  show = showInt
instance Show Bool where
  show = showBool
-- instance will be selected according to the argument type
show 1 ↪ showInt 1 ↪ "1"
show true ↪ showBool true ↪ "true"
```

- `show` can be defined as `showInt,,showBool`

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Overloaded Application

- Overloaded Application is standard.

  ```
  show : (Int -> String) & (Bool -> String)
  show = showInt,,showBool
  show 1 ↪ showInt 1 ↪ "1"
  show true ↪ showBool true ↪ "true"
  ```

- Adding overloading instances is simply by merging.

  ```
  newShow = show,,showDouble
  ```

Return type Overloading[7]

- Function implementation varies depending on the surrounding contexts.

[7] Koar Marntirosian et al. “Resolution as Intersection Subtyping via Modus Ponens”. In: *Proc. ACM Program. Lang.* 4.OOPSLA (2020).

Return type Overloading[7]

- Function implementation varies depending on the surrounding contexts.
- Consider Haskell's `read` function

```
read :: Read a => String -> a
instance Read Int where
  read = readInt
instance Read Bool where
  read = readBool
-- instance will be selected according to surrounding contexts
succ (read "1") ↪ succ (readInt "1") ↪ 2
not (read "true") ↪ succ (readBool "1") ↪ false
```

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-- instance will be selected according to surrounding contexts
succ (read "1") ↪ succ (readInt "1") ↪ 2
not (read "true") ↪ succ (readBool "1") ↪ false
```

- Calculi with merge operator can do in a similar way.

```
read = readInt,,readBool
```

[7] Koar Marntirosian et al. "Resolution as Intersection Subtyping via Modus Ponens". In: *Proc. ACM Program. Lang.* 4.OOPSLA (2020).

Nested Composition[8]

- It reflects *distributivity* of intersection types at the term level.

$$\{l : A\} \,\&\, \{l : B\} <: \{l : A \,\&\, B\} \quad \text{S-Distri-Rcd}$$

$$(A \rightarrow B) \,\&\, (A \rightarrow C) <: A \rightarrow (B \,\&\, C) \quad \text{S-Distri-Arr}$$

- Results extracted from <u>nested</u> terms will be <u>composed</u> when eliminating terms created by the merge operator.

[8] Xuan Bi, Bruno C. d. S. Oliveira, and Tom Schrijvers. "The essence of nested composition". In: *32nd European Conference on Object-Oriented Programming (ECOOP 2018)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik. 2018.

Nested Composition via Projection and Application

- For records

$$(\{x = e_1\},,\{x = e_2\}).x \hookrightarrow e_1,,e_2$$

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- For overloaded functions

$$f : Int \longrightarrow Int \longrightarrow Int$$
$$g : Int \longrightarrow Bool \longrightarrow Bool$$
$$(f,, g)\, 1 \hookrightarrow (f\, 1),, (g\, 1)$$

Nested Composition via Projection and Application

- For records

$$(\{x = e_1\} ,, \{x = e_2\}).x \hookrightarrow e_1 ,, e_2$$

- For overloaded functions

$$f : Int \to Int \to Int$$
$$g : Int \to Bool \to Bool$$
$$(f ,, g)\, 1 \hookrightarrow (f\, 1) ,, (g\, 1)$$

- Both cases are "unnatural"
  since we allow repeated labels and ambiguous overloaded application.

Goodness of Nested Composition

- *[Nested record composition]* Key feature of *Compositional Programming*[9].
    - solves the Expression Problem naturally.
    - models forms of family polymorphism.

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Goodness of Nested Composition

- *[Nested record composition]* Key feature of *Compositional Programming*[9].
  - solves the Expression Problem naturally.
  - models forms of family polymorphism.
- *[Nested function composition]* It enables *first-class curried overloaded functions*.
  - overloaded functions are default curried;
  - we can abstract and return overloaded functions in a flexible way;
  - it's a novel and interesting finding in this work.

[9] Weixin Zhang, Yaozhu Sun, and Bruno C. d. S. Oliveira. "Compositional Programming". In: *ACM Transactions on Programming Languages and Systems (TOPLAS)* 43.3 (2021), pp. 1–61.

Challenges in Type Inference

In traditional calculi, we have the following typing rule for application:

$$\frac{\Gamma \vdash e_1 \Rightarrow A \rightarrow B \qquad \Gamma \vdash e_2 \Leftarrow A}{\Gamma \vdash e_1\, e_2 \Rightarrow B} \text{ T-App}$$

This does not apply to case `show 1`, where

$$\frac{\Gamma \vdash show \Rightarrow A \,\&\, B \qquad \Gamma \vdash 1 \Leftarrow ?}{\Gamma \vdash show\, 1 \Rightarrow ?} \text{ T-App}$$

Challenges in Type Inference

A direct method is to:

1. assume we have the argument type A;
2. assume the type of function to be a intersection of function types:

$$(A_1 \longrightarrow B_1) \,\&\, (A_2 \longrightarrow B_2) \,\&\, \ldots \,\&\, (A_n \longrightarrow B_n)$$

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3. then iterate intersection types by comparing the argument type A and input type A_i;

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3. then iterate intersection types by comparing the argument type A and input type A_i;
4. compose the outputs as the result type

Challenges in Dynamic Semantics

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Challenges in Dynamic Semantics

A direct method is to:

1. assume the overloaded function to be a merge of functions,
2. then select correct instances according to the types.
    - call-by-value strategy
    - type-dependent semantics

Distributivity Breaks the Assumptions

```
pshow : Unit -> (Int -> String) & (Bool -> String)
pshow = λx. show
pshow unit 1 ↪ "1"
pshow unit true ↪ "true"
```

Distributivity Breaks the Assumptions

```
pshow : Unit -> (Int -> String) & (Bool -> String)
pshow = λx. show
pshow unit 1 ↪ "1"
pshow unit true ↪ "true"
```

- `pshow` is **not** a merge of functions (wrapped in a lambda);
- its type is **not** a intersection of function types;
- it's still treated as an overloaded function.

Re-interpret Subtyping

We can have two interpretations of $A <: B \rightarrow C$:

- Suppose A, B and C are given, we tell whether the subtyping holds.

$$(Int \rightarrow String) \,\&\, (Bool \rightarrow String) <: Int \rightarrow String$$

- Suppose A and B are given, we infer the result type C[10].

$$(Int \rightarrow String) \,\&\, (Bool \rightarrow String) <: Int \rightarrow ?$$

[10] which is also the type of overloaded application.

Applicative Subtyping

$A \ll S$ is a specialized subtyping used to infer the type of applications and projections [11].

$$
\begin{aligned}
A_1 \to A_2 \ll B &= A_2 && \textit{when } B <: A_1 && (1)\\
A_1 \to A_2 \ll B &= . && \textit{when } \neg(B <: A_1) && (2)\\
\{l = A\} \ll l &= A && && (3)\\
\{l_1 = A\} \ll l_2 &= . && \textit{when } l_1 \neq l_2 && (4)\\
A_1 \,\&\, A_2 \ll S &= (A_1 \ll S) \circledcirc (A_2 \ll S) && && (5)\\
A \ll S &= . && \textit{otherwise} && (6)
\end{aligned}
$$

[11] $S ::= A \mid l$, Selector S is either type A or label l

Examples of Applicative Subtyping

```
show 1
```

$$
\begin{aligned}
& (Int \to String) \,\&\, (Bool \to String) \ll Int \\
\text{by } (5) \hookrightarrow\ & (Int \to String) \ll Int \circledcirc (Bool \to String) \ll Int \\
\text{by } (1)\ (2) \hookrightarrow\ & String \circledcirc .
\end{aligned}
$$

```
read "1"
```

$$
\begin{aligned}
& (String \to Int) \,\&\, (String \to Bool) \ll String \\
\text{by } (5) \hookrightarrow\ & (String \to Int) \ll String \circledcirc (String \to Bool) \ll String \\
\text{by } (1) \hookrightarrow\ & Int \circledcirc Bool
\end{aligned}
$$

Composition Operators

One version that implements *nested composition semantics* [12].

$$
\begin{aligned}
\bullet \circledcirc \bullet &= \bullet \\
A_1 \circledcirc \bullet &= A_1 \\
\bullet \circledcirc A_2 &= A_2 \\
A_1 \circledcirc A_2 &= A_1 \,\&\, A_2
\end{aligned}
$$

[12] We have another version of the operator which models the overloading semantics

Examples (applying nested composition semantics)

$$
\begin{aligned}
(\mathit{Int} \to \mathit{String}) \,\&\, (\mathit{Bool} \to \mathit{String}) &\ll \mathit{Int} &&= \mathit{String} \\
(\mathit{String} \to \mathit{Int}) \,\&\, (\mathit{String} \to \mathit{Bool}) &\ll \mathit{String} &&= \mathit{Int} \,\&\, \mathit{Bool} \\
\{x : \mathit{String}\} \,\&\, \{y : \mathit{String}\} &\ll y &&= \mathit{String}
\end{aligned}
$$

Let arguments go "together"

We infer both the type of function (merges) and argument together and then compute.

$$\frac{\Gamma \vdash e_1 \Rightarrow A \qquad \Gamma \vdash e_2 \Rightarrow B \qquad \boxed{A \ll B = C}}{\Gamma \vdash e_1\, e_2 \Rightarrow C}\ \textsc{T-App}$$

Examples (applying nested composition semantics)

We assume Γ is $f : I \rightarrow I \rightarrow I, g : I \rightarrow B \rightarrow B$.[13]

$$\dfrac{\dfrac{\Gamma \vdash (f\,,,\,g) \Rightarrow (I \rightarrow I \rightarrow I) \,\&\, (I \rightarrow B \rightarrow B) \qquad \Gamma \vdash 2 \Rightarrow I}{\Gamma \vdash (f\,,,\,g)\; 2 \Rightarrow \boxed{(I \rightarrow I) \,\&\, (B \rightarrow B)}}\;\textsc{T-App} \qquad \Gamma \vdash \mathit{true} \Rightarrow B}{\Gamma \vdash (f\,,,\,g)\; 2\; \mathit{true} \Rightarrow \boxed{B}}\;\textsc{T-App}$$

1. $f\,,,\,g$
2. $(f\,,,\,g)\; 2$
3. $(f\,,,\,g)\; 2\; \mathit{true}$

[13] I stands for *Int*, B stands for *Bool*.

Metatheory

$$
\begin{aligned}
(Int \to String) \,\&\, (Bool \to String) &\ll Int &&= String \\
(String \to Int) \,\&\, (String \to Bool) &\ll String &&= Int \,\&\, Bool \\
\{x : String\} \,\&\, \{y : String\} &\ll y &&= String
\end{aligned}
$$

$$
\begin{aligned}
(Int \to String) \,\&\, (Bool \to String) &<: Int \to String \\
(String \to Int) \,\&\, (String \to Bool) &<: String \to Int \,\&\, Bool \\
\{x : String\} \,\&\, \{y : String\} &<: \{y : String\}
\end{aligned}
$$

Metatheory

Lemma (Soundness (Function))

If $A \ll B = C$, *then* $A <: B \to C$.

Lemma (Completeness (Function))

If $A <: B \to C$, *then* $\exists D, A \ll B = D \wedge D <: C$.

Calculi Syntax

$$
\begin{array}{ll}
\text{Expressions} & e ::= x \mid i \mid e : A \mid e_1\, e_2 \mid \lambda x\,.e : A \to B \mid e_1 ,, e_2 \mid \{l = e\} \mid e.l \\
\text{Raw Values} & p ::= i \mid \lambda x\,.e : A \to B \\
\text{Values} & v ::= \boxed{p : A^o \mid v_1 ,, v_2 \mid \{l = v\}} \\
\text{Contexts} & \Gamma ::= \cdot \mid \Gamma, x : A
\end{array}
$$

- Values carry extra annotations as runtime types;
- The dispatching is based on runtime types;
- The restriction on runtime types settles a canonical form of overloaded functions.

Operational Semantics

$$\textsc{Step-App}\quad \frac{(v_1 \bullet v_2) \hookrightarrow e}{v_1\, v_2 \longmapsto e}$$

$$\textsc{Step-Prj}\quad \frac{(v \bullet l) \hookrightarrow v'}{v.l \longmapsto v'}$$

Applicative Dispatching[14]

$\boxed{(v \bullet vl) \hookrightarrow e}$ *(Applicative Dispatching)*

$$\textsc{App-Lam}\quad \frac{v \longmapsto_A v'}{((\lambda x.\, e : A \to B) : C \to D \bullet v) \hookrightarrow e[x \mapsto v'] : D}$$

$$\textsc{App-Proj}\quad \frac{}{(\{l = v\} \bullet l) \hookrightarrow v}$$

$$\textsc{App-Mrg-L}\quad \frac{\langle v_2 \rangle \ll \langle vl \rangle = . \qquad (v_1 \bullet vl) \hookrightarrow e}{((v_1 \,,, v_2) \bullet vl) \hookrightarrow e}$$

$$\textsc{App-Mrg-R}\quad \frac{\langle v_1 \rangle \ll \langle vl \rangle = . \qquad (v_2 \bullet vl) \hookrightarrow e}{((v_1 \,,, v_2) \bullet vl) \hookrightarrow e}$$

$$\textsc{App-Mrg-P}\quad \frac{\langle v_1 \rangle \ll \langle vl \rangle \neq . \qquad \langle v_2 \rangle \ll \langle vl \rangle \neq . \qquad (v_1 \bullet vl) \hookrightarrow e_1 \qquad (v_2 \bullet vl) \hookrightarrow e_2}{((v_1 \,,, v_2) \bullet vl) \hookrightarrow e_1 \,,, e_2}$$

[14] $\langle v \rangle$ extracts the runtime type of v

Type Soundness and Determinism[15]

Theorem (Preservation)

If $\cdot \vdash e \Leftrightarrow A$ *and* $e \longmapsto e'$, *then* $\cdot \vdash e' \Leftarrow A$.

Theorem (Progress)

If $\cdot \vdash e \Leftrightarrow A$, *then* e *is a value or* $\exists e', e \longmapsto e'$.

Theorem (Determinism)

If e *is well-typed,* $e \longmapsto e_1$ *and* $e \longmapsto e_2$, *then* $e_1 = e_2$.

[15] held only in calculus with disjointness

Interpreter Implementation

- Statically typed;
- A dialect of Lisp;
- 382 Lines of Racket Code:
  - S-expression parsing included;
  - Contract-based runtime check;

Language Tour (1/3)

```
;; simple literals
42 42.2 #t #f

;; lambda abstraction
(λ (x : int) x int)

;; function application
((λ (x : int) x int) 1)
;; => (: 1 int)

;; annotate a "value" can force a downcast/upcast
(: (: 1 int)
   (& int int)) ;; => duplicate a number
;; => (m (: 1 int) (: 1 int))
```

Language Tour (2/3)

```
;; merge two values
(m 1 #t)

;; merge two functions
(m (λ (x : int) x int)
   (λ (x : bool) x bool))

;; merged function can be applied
((m (λ (x : int) x int)
    (λ (x : bool) x bool))
 1)
;; => (: 1 int)
```

Language Tour (3/3)

```
;; use int+ to add integers
(int+ 1 3)

;; use flo+ to add floats
(flo+ 1.0 2.1)

;; overload int+ and flo+ to create a polymorphic "double" function
((m (λ (x : int) (int+ x x) int)
    (λ (x : float) (flo+ x x) float))
 1)
;; => (: 2 int)
```

Conlusion

- Applicative Subtyping & Applicative Dispatching
  - Three Variants of Subtyping
  - Sound/Complete Lemmas
- Formalisation of Two Calculi Design
  - Type Sound Calculus with an Unrestricted Merge Operator
  - Deterministic Calculus with a Disjoint Merge Operator
- Coq Formalisation & Interpreter Implementation
  - https://github.com/juniorxxue/applicative-intersection

Future Work

- Application Mode
  - Alternative to Applicative Subtyping;
- Bidirectional Typing
  - Recover the advantage of check-mode;
- "Best-Match" Evaluation Strategy
- Compile to Racket
  - Static Type Checking and Resolution using Macro System