Applicative Intersection Types

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New languages keep being invented!

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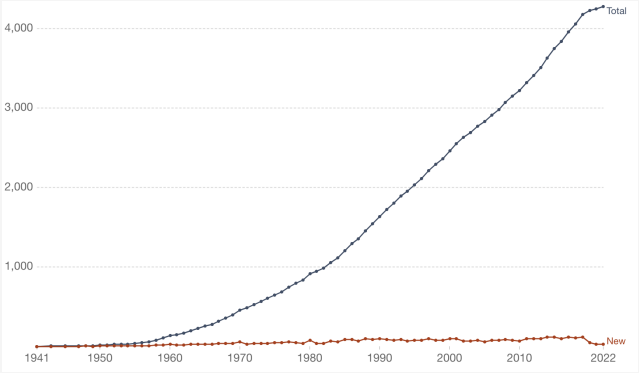

credit: pldb.com

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- New languages are prototyped by a small/core calculus;
- New features are often studied in an isolated environment;

And...

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And...

Features are not orthogonal;
Languages are not designed at once.

A general framework

contains wide features;
retains simplicity;
has extensibility;
and enjoys good properties,

A general framework

contains wide features;
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and enjoys good properties,

is desired by language designers and implementors.

Intersection Types is a nice t

> our goal is to use intersections and unions as general mechanisms for encoding language features, so we really should do it in full generality, or not at all..."[1]

[1] Jana Dun eld. Elaborating intersection and union types. Journal of Functional Programming 24 2 3 (2014), pp. 133 165.

Intersection Types

A term M having the type $A \& B$ means M has both A and B.

[2] Mario Coppo, Mariangiola Dezani-Ciancaglini, and Betti Venneri. Functional characters of solvable terms. Mathematical Logic Quarterly 27.2-6 (1981), pp. 45–58.

Intersection Types

A term M having the type $A \& B$ means M has both A and B.

Originally introduced by Coppo et al.[2], it allows $\lambda x.\, x\, x$ to be typed $((A \to B) \& A) \to B$.

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Intersection Types

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Originally introduced by Coppo et al.[2], it allows $x\,x$ to be typed $((A \to B) \& A) \to B$.

In languages like TypeScript, the intersection types are explicitly inhabitated.

```
interface Name { name: string; }
interface ID { id: number; }
type Person = Name & ID
let e : Person = { id: 42, name: 'Alice' };
```

[2] Mario Coppo, Mariangiola Dezani-Ciancaglini, and Betti Venneri. Functional characters of solvable terms. Mathematical Logic Quarterly 27.2-6 (1981), pp. 45–58.

Merge Operator[4]

$e_1 ; ; e_2$ means it can be used as e_1 or e_2.

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Merge Operator[4]

$e_1 ;; e_2$ means it can be used as e_1 or e_2.

Force intersection types to be explicitly introduced and inhabitated.

Typing for merge is[3]

$$\text{T-Mrg}\quad \frac{` e_1 ) \quad A \qquad ` e_2) \quad B}{` e_1 ;; e_2) \quad A \& B}$$

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Merge operator adds expressive power and enables many applications

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Extensible Records[5]

Records can be represented by syntactic sugar of merge operator

$\{x = e_1, y = e_2, z = e_3\}$ can be viewed as $\{x = e_1\} , \{y = e_2\} , \{z = e_3\}$

[5] Luca Cardelli and John C Mitchell. Operations on records. Mathematical structures in computer science 1.1 (1991), pp. 3–48.

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Record width subtyping for free

$$\{l_i : T_i\}^{i=1::n::n+k} <: \{l_i : T_i\}^{1::n}$$

is subsumed by

$$\{l_1 : A\} \& \{l_2 : B\} <: \{l_1 : A\}$$

is subsumed by

$$A \& B <: A$$

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Record Projection

Record Projection is standard.

$$(\{x = e_1\} \mathbin{;} \{y = e_2\}).x \longrightarrow e_1$$

$$(\{x = e_1\} \mathbin{;} \{y = e_2\}).y \longrightarrow e_2$$

Record Concatenation is simply merging.

$$(\{x = e_1\} \mathbin{;} \{y = e_2\}) \mathbin{;} \{z = e_3\}$$

Overloaded Functions[6]

Function implementation varies depending on the types of arguments.

[6] Giuseppe Castagna, Giorgio Ghelli, and Giuseppe Longo. A calculus for overloaded functions with subtyping Information and Computation 117.1 (1995), pp. 115–135.

Overloaded Functions[6]

Function implementation varies depending on the types of arguments.

Consider Haskell's `show` function.

```
show :: Show a => a -> String
instance Show Int where
  show = showInt
instance Show Bool where
  show = showBool
-- instance will be selected according to the argument type
show 1 ⇒ showInt 1 ⇒ "1"
show true ⇒ showBool true ⇒ "true"
```

`show` can be defined as `showInt ,, showBool`

[6] Giuseppe Castagna, Giorgio Ghelli, and Giuseppe Longo. A calculus for overloaded functions with subtyping. Information and Computation 117.1 (1995), pp. 115–135.

Overloaded Application

Overloaded Application is standard.

```
show : (Int -> String) & (Bool -> String)
show = showInt,,showBool
show 1 ¦ showInt 1 ¦ "1"
show true ¦ showBool true ¦ "true"
```

Adding overloading instances is simply by merging.

```
newShow = show,,showDouble
```

Return type Overloading[7]

Function implementation varies depending on the surrounding contexts.

[7] Koar Marntirosian et al. Resolution as Intersection Subtyping via Modus Ponens. In Proc. ACM Program. Lang. 4.OOPSLA (2020).

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Consider Haskell's read function

```
read :: Read a => String -> a
instance Read Int where
  read = readInt
instance Read Bool where
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-- instance will be selected according to surrounding contexts
succ (read "1")   ⇝  succ (readInt "1")   ⇝  2
not (read "true")   ⇝  succ (readBool "1")   ⇝  false
```

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-- instance will be selected according to surrounding contexts
succ (read "1")  ⁏  succ (readInt "1")  ⁏  2
not (read "true")  ⁏  succ (readBool "1")  ⁏  false
```

Calculi with merge operator can do in a similar way.

```
read = readInt,,readBool
```

[7] Koar Marntirosian et al. Resolution as Intersection Subtyping via Modus Ponens. In Proc. ACM Program. Lang. 4.OOPSLA (2020).

Nested Composition[8]

It reflects distributivity of intersection types at the term level.

$$\{l : A\} \& \{l : B\} <: \{l : A \& B\} \quad \text{S-Distri-Rcd}$$

$$(A \to B) \& (A \to C) <: A \to (B \& C) \quad \text{S-Distri-Arr}$$

Results extracted from nested terms will be composed when eliminating terms created by the merge operator.

[8] Xuan Bi, Bruno C. d. S. Oliveira, and Tom Schrijvers. The essence of nested composition. In 32nd European Conference on Object-Oriented Programming (ECOOP 2018). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik. 2018.

Nested Composition via Projection and Application

For records

$$(\{x = e_1\} ; \{x = e_2\}).x \not\to e_1 ; e_2$$

Nested Composition via Projection and Application

For records

$$(\{x = e_1\} ; \{x = e_2\}) : x \to e_1 ; e_2$$

For overloaded functions

$$f : \mathrm{Int} \to \mathrm{Int} \to \mathrm{Int}$$
$$g : \mathrm{Int} \to \mathrm{Bool} \to \mathrm{Bool}$$
$$(f ; g)\ 1 \to (f\ 1) ; (g\ 1)$$

Nested Composition via Projection and Application

For records

$$(\{x = e_1\} ,, \{x = e_2\}).x \longrightarrow e_1 ,, e_2$$

For overloaded functions

$$f : \mathrm{Int} \to \mathrm{Int} \to \mathrm{Int}$$
$$g : \mathrm{Int} \to \mathrm{Bool} \to \mathrm{Bool}$$
$$(f ,, g)\ 1 \longrightarrow (f\ 1) ,, (g\ 1)$$

Both cases are "unnatural"
since we allow repeated labels and ambiguous overloaded application

Goodness of Nested Composition

[Nested record composition] Key feature of Compositional Programming[9]

- solves the Expression Problem naturally.
- models forms of family polymorphism.

[9] Weixin Zhang, Yaozhu Sun, and Bruno C. d. S. Oliveira. Compositional Programming. In: ACM Transactions on Programming Languages and Systems (TOPLAS) 43.1 (2021), pp. 1–61.

Goodness of Nested Composition

[Nested record composition] Key feature of Compositional Programming[9]

- solves the Expression Problem naturally.
- models forms of family polymorphism.

[Nested function composition] It enablesrst-class curried overloaded functions.

- overloaded functions are default curried;
- we can abstract and return overloaded functions in a exible way;
- it's a novel and interesting nding in this work.

[9] Weixin Zhang, Yaozhu Sun, and Bruno C. d. S. Oliveira. Compositional Programming. In: ACM Transactions on Programming Languages and Systems (TOPLAS) 43.1 (2021), pp. 1 61.

Challenges in Type Inference

In traditional calculi, we have the following typing rule for application:

$$\frac{\Gamma \vdash e_1 : A \to B \qquad \Gamma \vdash e_2 : A}{\Gamma \vdash e_1\, e_2 : B}\ \text{T-App}$$

This does not apply to cases like $show\ 1$, where

$$\frac{\Gamma \vdash show : A \& B \to \ldots \qquad \Gamma \vdash 1 : ?}{\Gamma \vdash show\ 1 : \ldots}\ \text{T-App}$$

Challenges in Type Inference

A direct method is to:

1. assume we have the argument type A
2. assume the type of function to be a intersection of function types:

$$(A_1 \to B_1) \& (A_2 \to B_2) \& \cdots \& (A_n \to B_n)$$

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3. then iterate intersection types by comparing the argument type A and input type A_i;
4. compose the outputs as the result type

Challenges in Dynamic Semantics

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2. then select correct instances according to the types.

    call-by-value strategy
    type-dependent semantics

Distributivity Breaks the Assumptions

```
pshow : Unit -> (Int -> String) & (Bool -> String)
pshow = x. show
pshow unit 1 ¦ "1"
pshow unit true ¦ "true"
```

Distributivity Breaks the Assumptions

```
pshow : Unit -> (Int -> String) & (Bool -> String)
pshow = x. show
pshow unit 1 ;! "1"
pshow unit true ;! "true"
```

- `pshow` is not a merge of functions (wrapped in a lambda);
- its type is not a intersection of function types;
- it's still treated as an overloaded function.

Re-interpret Subtyping

We can have two interpretations of $A <: B \to C$.

Suppose A, B and C are given, we tell whether the subtyping holds.

$$(\mathrm{Int} \to \mathrm{String}) \& (\mathrm{Bool} \to \mathrm{String}) <: \mathrm{Int} \to \mathrm{String}$$

Suppose A and B are given, we infer the result type C.[10]

$$(\mathrm{Int} \to \mathrm{String}) \& (\mathrm{Bool} \to \mathrm{String}) <: \mathrm{Int} \to ?$$

[10] which is also the type of overloaded application.

Applicative Subtyping

A S is a specialized subtyping used to infer the type of applications and projections[11]

$$A_1 \to A_2 \quad B = A_2 \qquad \text{when } B <: A_1 \qquad (1)$$

$$A_1 \to A_2 \quad B = \bot \qquad \text{when } \neg (B <: A_1) \qquad (2)$$

$$\{l = A\} \quad l = A \qquad (3)$$

$$\{l_1 = A\} \quad l_2 = \bot \qquad \text{when } l_1 \neq l_2 \qquad (4)$$

$$A_1 \& A_2 \quad S = (A_1 \quad S) \& (A_2 \quad S) \qquad (5)$$

$$A \quad S = \bot \qquad \text{otherwise} \qquad (6)$$

[11] $S ::= A \mid l$, Selector S is either type A or label l

Examples of Applicative Subtyping

show 1

$$\begin{array}{l} \quad (\mathsf{Int} \to \mathsf{String}) \,\&\, (\mathsf{Bool} \to \mathsf{String}) \quad \mathsf{Int} \\ \mathsf{by}(5) \;\dagger\; (\mathsf{Int} \to \mathsf{String}) \quad \mathsf{Int} \;\}\; (\mathsf{Bool} \to \mathsf{String}) \quad \mathsf{Int} \\ \mathsf{by}(1)\,(2) \;\dagger\; \mathsf{String}\} \;: \end{array}$$

read "1"

$$\begin{array}{l} \quad (\mathsf{String} \to \mathsf{Int}) \,\&\, (\mathsf{String} \to \mathsf{Bool}) \quad \mathsf{String} \\ \mathsf{by}(5) \;\dagger\; (\mathsf{String} \to \mathsf{Int}) \quad \mathsf{String}\} \;(\mathsf{String} \to \mathsf{Bool}) \quad \mathsf{String} \\ \mathsf{by}(1) \;\dagger\; \mathsf{Int}\;\} \;\mathsf{Bool} \end{array}$$

Composition Operators

One version that implements nested composition semantics[12].

$$
\begin{aligned}
: \} \ : &= : \\
A_1 \} \ : &= A_1 \\
: \} \ A_2 &= A_2 \\
A_1 \} \ A_2 &= A_1 \& A_2
\end{aligned}
$$

[13] We have another version of the operator which models the overloading semantics

Examples (applying nested composition semantics)

$$
\begin{array}{lll}
(\mathsf{Int} \to \mathsf{String}) \,\&\, (\mathsf{Bool} \to \mathsf{String}) & \mathsf{Int} & = \mathsf{String} \\
(\mathsf{String} \to \mathsf{Int}) \,\&\, (\mathsf{String} \to \mathsf{Bool}) & \mathsf{String} & = \mathsf{Int} \,\&\, \mathsf{Bool} \\
\{x : \mathsf{String}\} \,\&\, \{y : \mathsf{String}\} & y & = \mathsf{String}
\end{array}
$$

Let arguments go "together"

We infer both the type of function (merges) and argument together and then compute.

$$\frac{\grave{}\ e_1) \quad A \qquad \grave{}\ e_2) \quad B \qquad A \quad B = C}{\grave{}\ e_1 e_2) \quad C}\ \text{T-App}$$

Examples (applying nested composition semantics)

We assume is $f : I \to I \to I; g : I \to B \to B$.[13]

$$\dfrac{\dfrac{\vdash (f ;; g)\ 2 : (I \to I \to I) \,\&\, (I \to B \to B) \qquad \vdash 2 : I}{\vdash (f ;; g)\ 2 : (I \to B \to B)}\ \text{T-App} \qquad \vdash \text{true} : B}{\vdash (f ;; g)\ 2\ \text{true} : B}\ \text{T-App}$$

1. $f ;; g$
2. $(f ;; g)\ 2$
3. $(f ;; g)\ 2\ \text{true}$

[13] I stands for Int, B stands for Bool

Metatheory

$$(\mathsf{Int} \to \mathsf{String}) \,\&\, (\mathsf{Bool} \to \mathsf{String}) \quad \mathsf{Int} = \mathsf{String}$$
$$(\mathsf{String} \to \mathsf{Int}) \,\&\, (\mathsf{String} \to \mathsf{Bool}) \quad \mathsf{String} = \mathsf{Int} \,\&\, \mathsf{Bool}$$
$$\{x : \mathsf{String}\} \,\&\, \{y : \mathsf{String}\} \quad y = \mathsf{String}$$

$$(\mathsf{Int} \to \mathsf{String}) \,\&\, (\mathsf{Bool} \to \mathsf{String}) <: \mathsf{Int} \to \mathsf{String}$$
$$(\mathsf{String} \to \mathsf{Int}) \,\&\, (\mathsf{String} \to \mathsf{Bool}) <: \mathsf{String} \to \mathsf{Int} \,\&\, \mathsf{Bool}$$
$$\{x : \mathsf{String}\} \,\&\, \{y : \mathsf{String}\} <: \{y : \mathsf{String}\}$$

Metatheory

Lemma (Soundness (Function))
If $A \; B = C$, then $A <: B \to C$.

Lemma (Completeness (Function))
If $A <: B \to C$, then $\exists D$, $A \; B = D \wedge D <: C$.

Calculi Syntax

Expressions	$e ::= x \mid i \mid j \mid e : A \mid e_1\, e_2 \mid \lambda x : e : A \,!\, B \mid e_1; ; e_2 \mid \{ l = e \mid j \mid e : l$
Raw Values	$p ::= i \mid j \mid \lambda x : e : A \,!\, B$
Values	$v ::= p : A^{\circ} \mid j \mid v_1; ; v_2 \mid \{ l = v \mid g$
Contexts	$::= \square \mid j \mid ; x : A$

Values carry extra annotations as runtime types;

The dispatching is based on runtime types;

The restriction on runtime types settles a canonical form of overloaded functions.

Operational Semantics

$$\textbf{Step-App}\quad \frac{(v_1 \quad v_2) \;\dot{,}\; e}{v_1 v_2 \;\; 7! \;\; e}$$

$$\textbf{Step-Prj}\quad \frac{(v \quad l) \;\dot{,}\; v^0}{v.l \;\; 7! \;\; v^0}$$

Applicative Dispatching[14]

$$\boxed{(v\ \ vl)\ \dot{!}\ \ e}$$

(Applicative Dispatching)

App-Lam

$$\frac{v\ 7!\ _{A}\ v^{0}}{((\ x{:}\,e{:}\,A\,!\ \ B){:}\,C\,!\ \ D\ \ v)\ \dot{!}\ \ e[x\ 7\ v^{0}]{:}\,D}$$

App-Proj

$$\frac{}{(fl = vg\ \ l)\ \dot{!}\ \ v}$$

App-Mrg-L

$$\frac{\langle v_2\rangle\ \ h\ \ vli = :\ \ (v_1\ \ vl)\ \dot{!}\ \ e}{((v_1;;\,v_2)\ \ vl)\ \dot{!}\ \ e}$$

App-Mrg-R

$$\frac{\langle v_1\rangle\ \ h\ \ vli = :\ \ (v_2\ \ vl)\ \dot{!}\ \ e}{((v_1;;\,v_2)\ \ vl)\ \dot{!}\ \ e}$$

App-Mrg-P

$$\frac{\langle v_1\rangle\ \ h\ \ vli\ 6{=}\ :\ \ \langle v_2\rangle\ \ h\ \ vli\ 6{=}\ :\ \ (v_1\ \ vl)\ \dot{!}\ \ e_1\ \ \ (v_2\ \ vl)\ \dot{!}\ \ e_2}{((v_1;;\,v_2)\ \ vl)\ \dot{!}\ \ e_1;;\,e_2}$$

[14] $\langle v\rangle$ extracts the runtime type of v

Type Soundness and Determinism[15]

Theorem (Preservation)

If $\vdash e, A$ and $e \mapsto e'$, then $\vdash e' \subset A$.

Theorem (Progress)

If $\vdash e, A$, then e is a value or, $e \mapsto e'$.

Theorem (Determinism)

If e is well-typed, $e \mapsto e_1$ and $e \mapsto e_2$, then $e_1 = e_2$.

[15] held only in calculus with disjointness

Interpreter Implementation

Statically typed;

A dialect of Lisp;

382 Lines of Racket Code:

- S-expression parsing included;
- Contract-based runtime check;

Language Tour (1/3)

```
;; simple literals
42 42.2 #t #f

;; lambda abstraction
(   (x : int) x int)

;; function application
((    (x : int) x int) 1)
;; => (: 1 int)

;; annotate a "value" can force a downcast/upcast
(: (: 1 int)
   (& int int))      ;; => duplicate a number
;; => (m (: 1 int) (: 1 int))
```

Language Tour (2/3)

```
;; merge two values
(m 1 #t)

;; merge two functions
(m (  (x : int) x int)
   (  (x : bool) x bool))

;; merged function can be applied
((m (  (x : int) x int)
    (  (x : bool) x bool))
 1)
;; => (: 1 int)
```

Language Tour (3/3)

```
;; use int+ to add integers
(int+ 1 3)

;; use flo+ to add floats
(flo+ 1.0 2.1)

;; overload int+ and flo+ to create a polymorphic "double" function
((m (   (x : int) (int+ x x) int)
    (   (x : float) (flo+ x x) float))
 1)
;; => (: 2 int)
```

Conlusion

- Applicative Subtyping & Applicative Dispatching
  - Three Variants of Subtyping
  - Sound/Complete Lemmas
- Formalisation of Two Calculi Design
  - Type Sound Calculus with an Unrestricted Merge Operator
  - Deterministic Calculus with a Disjoint Merge Operator
- Coq Formalisation & Interpreter Implementation
  - https://github.com/juniorxxue/applicative-intersection

Future Work

Application Mode

- Alternative to Applicative Subtyping;

Bidirectional Typing

- Recover the advantage of check-mode;

“Best-Match" Evaluation Strategy

Compile to Racket

- Static Type Checking and Resolution using Macro System